RESEARCH STATEMENT

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My main research interests are: discrete harmonic analysis, analytic number theory, and moments of arithmetic functions. I have completed research projects in ergodic theorem, primes distribution, partitions theory, and additive number theory. My notable results during my Ph.D. are ℓ^p-improving inequalities of several averages over prime numbers, reporting a cancellation phenomenon as well as its application in partition theory and distribution of prime numbers, and solving a statistical version of the 105 problem raised by Erdős and Graham.

1. Summary of My Results

1.1. **The Erdős-Graham open problem.** Erdős, Graham, Ruzsa, and Straus in [1] proved the following theorem:

Theorem 1.1. *For primes p, q, there are infinitely many integers n such that* $\gcd\left(\binom{2n}{n}, pq\right) = 1$.

The next natural step is an open problem by Erdős-Graham-Pomerance.

Question 1.2. [2, The 105 problem] *Are there infinitely many integers n such that* $\gcd\left(\binom{2n}{n}, 105\right) = 1$*?*

A sufficient condition for $\gcd\left(p, \binom{2n}{n}\right) = 1$ is that when computing $n+n$ base p, there are no carries. That is writing the digits of n base p as $n_{p,1}, n_{p,2}, \cdots$, one has $n_{p,j} < \lfloor p/2 \rfloor$ for all j. In this light, the Erdos-Graham conjecture would be true if there are infinitely many integers with $n_{3,j} < 2$, $n_{5,j} < 3$, and $n_{7,j} < 4$ for all j. What we proved was a statistical case of a generalization of the 105 problem:

Theorem 1.3. *Let $p_1 < p_2 \cdots < p_r$ are primes larger than a univeral constant. There are infinitely many integers n such that*

$$v_{p_i}\binom{2n}{n} = o_{p_i,r}\left(\log_{p_i}(n)\right) \text{ for } 1 \leqslant i \leqslant r$$

where v_p is the valuation function. Note the little oh is just with respect to p_i, r and not n.

The idea is to pick a block of digits with length $k = o(\log\log\log\log(n))$ in a base prime p and fix the digits of blocks in other bases p_i. It turns out that to perform this algorithm, we need to solve the following problem:

Theorem 1.4. *Let N be a large integer and p be a fixed prime number. For $1 \leqslant i \leqslant r$, define $\alpha_{i,j} = p_i^{\{j\frac{\log(p)}{\log(p_i)}\}-1}$. For all but at most ϵN integers $\ell \leqslant N$ the following holds: for arbitrary $\beta_1, \beta_2, \dots \in [0,1)$, there exists $n \leqslant p^{\epsilon k}$, such that $n\alpha_{i,\ell} + \beta_i$ have small digits base p_i.*

If we have Schanuel's conjecture, the terms $\{\frac{\log(p)}{\log(p_i)}\}$ become independent over $\mathbb{Q}$, which makes the theorem easy to solve. The way to bypass Schanuel's conjecture was to parametrize all the curves from the dependency and use similar ideas like Gower's methods to find a large enough uniform distribution in our construction. We needed to show a certain matrix has a positive determinant.

1.2. **The cancellation phenomenon.** What we have discovered is that the Pentagonal number theorem is just the tip of the iceberg, and that there is a very general class of sums like this that are small - much smaller than one would guess based on a probabilistic heuristic. Roughly, we proved that for certain real number $c > 0$ there exist $w_c < 1$ such that

$$\sum_{f(n) \leqslant x} (-1)^n e^{c\sqrt{x - f(n)}} = e^{cw_c\sqrt{x}},$$

where f is a quadratic polynomial with positive leading coefficient. More precisely, we proved two sets of cancellation inequalities depending on wether c is a real or complex number. We count three applications for the cancellation theorems. First, we showed that in the "Weak pentagonal number theorem," we can replace the partition function $p(n)$ with Chebyshev Ψ function. A weak version of what we proved can be written as

$$\Psi(e^{\sqrt{x}}) = 2 \sum_{0<\ell<\sqrt{xT}/2} \left(\Psi\left(e^{\sqrt{x-\frac{(2\ell-1)^2}{T}}}\right) - \Psi\left(e^{\sqrt{x-\frac{(2\ell)^2}{T}}}\right)\right) + O\left(e^{(\frac{5}{6}+\epsilon)\sqrt{x}}\right) \quad \text{where } T := e^{\frac{2\sqrt{x}}{3}}.$$

Second we got a generalization of the Pentagonal Number Theorem for several forms of the famous partitions. For example for the usual number of partitions $p(n)$ we have

$$\sum_{\ell^2<x} (-1)^\ell p(x-\ell^2) \sim 2^{-3/4} x^{-1/4} \sqrt{p(x)}.$$

In another application of our cancellation theorem, we gave a constructive answer to the approximation of Prouhet-Tarry-Escott problem.

Corollary 1.5. *The approximation version of the Prouhet-Tarry-Escott problem has a constructive solution for $k \sim \frac{n^{6/7}}{\log(n)}$ and $N = 2n^2$ as follows. For $1 \leqslant i \leqslant n$*

$$a_i = 2n^2 - (2i-2)^2 \in \mathbb{N} \quad \textit{and} \quad b_i = 2n^2 - (2i-1)^2 \in \mathbb{N}.$$

That is,

$$\sum_{i=1}^{n} a_i^r - b_i^r \ll n^r.$$

1.3. Improving Inequality on Average over primes.

Assume the following average along the primes.

$$A_N f(x) := \frac{1}{N} \sum_{n<N} f(x-n)\Lambda(n).$$

The set of primes is "full dimensional", so one can see that in an appropriate sense, an ℓ^1 function is improved to an ℓ^∞ function. The precise result is

$$\| A_N f \|_\infty \ll N^{-1} \| f \|_1 \log^t (\| f \|_1).$$

where $f \in L^1(\mu)$ is a positive function, and $t = 1$ if we assume the GRH, otherwise $t = 2$. The point is that the bound is scale invariant. The argument entails some subtle variants of the approach initiated by Bourgain. In particular, the notion of smooth numbers is essential due to their multiplicative properties, which helps us better control the Ramanujan sums in the low part. Moreover, in the absence of GRH, a systematic analysis of the exceptional Dirichlet's characters comes into the proof.

2. Future Directions

I will add two of the possible future directions from my results.

2.1. The Cancellation Result.

Numerical results suggest that

$$\sum_{\ell^2<x} (-1)^\ell e^{\sqrt{x-\ell^2}} = e^{o(\sqrt{x})}.$$

I am interested in studying this sum to see if we can get a better result. I also like to work on the Prouhet-Tarry-Escott problem to find a constructive solution at least for the range $k \sim n^\epsilon$. Having this result helps to get a better view of the distribution of smooth numbers in short intervals. In the best case scenario, it can even solve cases of the Vinogradov residue theorem. Dr. Soundarajan has many different results involving smooth numbers, and he can help me get the best result I could in this direction.

2.2. **Averages along primes in a number field.** Before mentioning that result, I am also interested in studying the following averages, which could be one of the natural next steps after the new bilinear breakthrough by Tao, Mirek, and Krause

$$A_N fg(x) := \frac{1}{N} \sum_{n<N} \Lambda(n) f(x-n) g(x-n^2).$$

I also like to interested in proving a quantitative inequality for

$$A_N f(x) = \frac{1}{N^2} \sum_{N(n_1), N(n_2)<N} \Lambda_{\mathbb{F}}(n_1) \Lambda_{\mathbb{F}}(n_2) f(x-n_1-n_2)$$

where $\mathbb{F}$ is a quadratic number field with class number 1. Note that this result falls short of the Goldbach conjecture in such rings of integers. We have a good chance of proving at least an average version of a result for Gaussian integers (with more logarithm weight functions) because it is known that Gaussian primes have uniform distribution both in magnitudes and in-phase. I believe that Dr. Soundarajan is one of the best people I can work with because of his experience with Dirichlet characters and the theory of $L-$functions in quadratic number fields.

References

[1] Erdös, P., et al. "On the Prime Factors of 2nn." Mathematics of Computation (1975): 83-92.

[2] Pomerance, Carl. "Divisors of the middle binomial coefficient." The American Mathematical Monthly 122.7 (2015): 636-644.

[3] Croot, Ernie, and Hamed Mousavi. "On a Class of Sums with Unexpectedly High Cancellation, and its Applications." arXiv preprint arXiv:1909.12470 (2019).

[4] Lacey, Michael T., Hamed Mousavi, and Yaghoub Rahimi. "Endpoint ℓ^r improving estimates for Prime averages." arXiv preprint arXiv:2101.10401 (2021).

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